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# Generalized polarizabilities and the spin-averaged amplitude in virtual Compton scattering off the nucleon

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## Abstract

We discuss the low-energy behavior of the spin-averaged amplitude of virtual Compton scattering (VCS) off a nucleon. Based on gauge invariance, Lorentz invariance and the discrete symmetries, it is shown that to first order in the frequency of the final real photon only two generalized polarizabilities appear. Different low-energy expansion schemes are discussed and put into perspective.

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## I. INTRODUCTION

Virtual Compton scattering (VCS) off the proton, as tested in, e.g., the reaction  $e^- + p \rightarrow e^- + p + \gamma$ , has recently attracted considerable interest [1]. Several experiments have been proposed [2–6], utilizing the opportunities which a virtual spacelike photon offers, namely, an additional longitudinal polarization degree of freedom and the fact that energy and momentum transfer of the virtual photon can be varied independently. At the same time, in comparison with real Compton scattering, the extraction of new experimental information will be more difficult since the process  $e^- + p \rightarrow e^- + p + \gamma$  contains an interference between VCS and the Bethe-Heitler contribution, describing radiation off the electron. On the theoretical side, the low-energy theorem (LET) of Low [7] and Gell-Mann and Goldberger [8] has lately been extended to include virtual photons as well [9,10]. The structure-dependent part beyond the LET was analyzed in [9] in terms of a multipole expansion. Keeping only terms linear in the energy of the final real photon, the model-dependent amplitude was parametrized in terms of ten “generalized polarizabilities”, and these polarizabilities were evaluated in the framework of a nonrelativistic quark model [9,11]. Predictions for the spin-averaged polarizabilities  $\alpha(|\vec{q}|)$  and  $\beta(|\vec{q}|)$  were obtained by several authors within various frameworks, such as an effective Lagrangian including resonances and  $t$ -channel exchanges [12], the linear  $\sigma$  model [13], and the heavy-baryon formulation of chiral perturbation theory [14]. An alternative low-energy expansion for virtual Compton scattering off a spin-zero target, and thus implicitly also for the spin-averaged part of the nucleon VCS amplitude, has been obtained in [15].

In [13] an interesting observation was made: within the framework of the linear  $\sigma$  model only two of the three scalar generalized polarizabilities introduced in [9] were found to be independent. In the following, we will reinvestigate the spin-independent part of the VCS amplitude and demonstrate that the findings of [13] can be proven to be a general consequence of charge-conjugation symmetry combined with crossing symmetry. Unless charge-conjugation invariance is violated, there are only two independent scalar generalized polarizabilities. Furthermore, we illustrate how the standard limit of real Compton scattering is naturally obtained, if the expansion in the final-photon energy is not truncated at first order. Finally, the low-energy expansion of [15] and its application in the framework of a heavy-baryon calculation [14] are connected to this work.

## II. GENERAL FORMALISM

For the purpose of simplicity, we consider the VCS amplitude for a spinless target, e.g., a positively charged pion:  $\gamma^*(q, \epsilon) + \pi^+(p) \rightarrow \gamma^*(q', \epsilon') + \pi^+(p')$ . In the following discussion, we will thus always refer to the pion, but the general results also apply to the spin-averaged amplitude of VCS off the proton [16,17], which is, of course, the reaction of current experimental and theoretical interest [1].

Using the conventions of Bjorken and Drell [18], the invariant amplitude may be written as

$$\mathcal{M} = -2Mie^2\epsilon_\mu\epsilon'_\nu M^{\mu\nu}, \quad (1)$$

where  $\epsilon$  and  $\epsilon'$  denote “polarization vectors” of the initial and final photon, respectively,  $e > 0$  is the elementary charge ( $e^2/4\pi = 1/137$ ), and  $M$  is the mass of the target, here the pion. According to [18], the normalization of the respective invariant amplitude  $\mathcal{M}$  differs by a factor of  $2M$  between the pion and proton case. When considering VCS off the proton, one therefore has to omit this factor in Eq. (1) in order to obtain the same normalization of the Compton tensor  $M^{\mu\nu}$  as for the pion case. In this section we still allow both photons to be virtual, and only in the following section we will restrict ourselves to the application we are interested in, namely,  $e^- + \pi^+ \rightarrow e^- + \pi^+ + \gamma$ .

We split the total VCS tensor into two parts  $A$  and  $B$  [8], where class  $A$  contains the pole terms, possibly together with some appropriate piece to ensure gauge invariance, and class  $B$  contains the rest,

$$M^{\mu\nu} = M_A^{\mu\nu} + M_B^{\mu\nu}. \quad (2)$$

We assume that the division into  $A$  and  $B$  was done in such a fashion that all symmetry principles are individually satisfied by  $M_A^{\mu\nu}$  and  $M_B^{\mu\nu}$ . With the above separation,  $M_B^{\mu\nu}$  is by construction regular as  $q^\mu \rightarrow 0$  or  $q'^\mu \rightarrow 0$ . In fact, there is some degree of arbitrariness concerning which contribution is included into class  $A$ . Different choices will differ by separately gauge invariant, regular terms (see [10,15] for more details).

We will now discuss a few general properties of  $M_B^{\mu\nu}$ . Under Lorentz transformations  $M_B^{\mu\nu}$  transforms as a proper second-rank Lorentz tensor which can be constructed in terms of  $q^\mu$ ,  $q'^\mu$ , and  $P^\mu = p^\mu + p'^\mu$ . A complete set of independent tensors is given by

$$g^{\mu\nu}, P^\mu P^\nu, P^\mu q^\nu, q^\mu P^\nu, P^\mu q'^\nu, q'^\mu P^\nu, q^\mu q^\nu, q'^\mu q'^\nu, q^\mu q'^\nu, q'^\mu q^\nu. \quad (3)$$

These tensors are multiplied by scalar functions of the invariants available, e.g.,  $q^2$ ,  $q'^2$ ,  $q \cdot q'$ , and  $q \cdot P = q' \cdot P$ . Symmetry with respect to charge conjugation implies that the VCS tensor is the same for both  $\pi^+$  and  $\pi^-$ ,

$$M_{B,\pi^+}^{\mu\nu}(p', q'; p, q) = M_{B,\pi^-}^{\mu\nu}(p', q'; p, q), \quad (4)$$

which can be converted into a constraint involving, say,  $M_{B,\pi^+}^{\mu\nu}$  only, by making use of pion crossing (see, e.g., [19]),

$$M_{B,\pi^+}^{\mu\nu}(p', q'; p, q) = M_{B,\pi^+}^{\mu\nu}(-p, q'; -p', q), \quad (5)$$

yielding finally

$$M_B^{\mu\nu}(q, q', P) = M_B^{\mu\nu}(q, q', -P), \quad (6)$$

where from now on we omit the subscript  $\pi^+$  and, using four-momentum conservation, express  $M_B^{\mu\nu}$  as a function of the three independent momenta  $q$ ,  $q'$ , and  $P$ . In order to easily implement the constraints due to photon-crossing symmetry,

$$M_B^{\mu\nu}(q, q', P) = M_B^{\nu\mu}(-q', -q, P), \quad (7)$$

and the combination of charge-conjugation symmetry with pion-crossing symmetry, Eq. (6), we choose the following parametrization of  $M_B^{\mu\nu}$  [15,16]:

$$\begin{aligned}
M_B^{\mu\nu}(q, q', P) = & A g^{\mu\nu} + B P^\mu P^\nu + C (P^\mu q^\nu - q'^\mu P^\nu) + \tilde{C} (P^\mu q^\nu + q'^\mu P^\nu) \\
& + D (P^\mu q'^\nu - q^\mu P^\nu) + \tilde{D} (P^\mu q'^\nu + q^\mu P^\nu) + E (q^\mu q^\nu + q'^\mu q'^\nu) \\
& + \tilde{E} (q^\mu q^\nu - q'^\mu q'^\nu) + F q^\mu q'^\nu + G q'^\mu q^\nu,
\end{aligned} \tag{8}$$

where the scalar functions have the following properties:

$$f(q^2, q'^2, q \cdot q', q \cdot P) = +f(q'^2, q^2, q \cdot q', -q \cdot P), \quad \text{for } f = A, B, C, D, E, F, G, \tag{9}$$

$$f(q^2, q'^2, q \cdot q', q \cdot P) = -f(q'^2, q^2, q \cdot q', -q \cdot P), \quad \text{for } f = \tilde{C}, \tilde{D}, \tilde{E}, \tag{10}$$

$$f(q^2, q'^2, q \cdot q', q \cdot P) = +f(q^2, q'^2, q \cdot q', -q \cdot P), \quad \text{for } f = A, B, E, \tilde{E}, F, G, \tag{11}$$

$$f(q^2, q'^2, q \cdot q', q \cdot P) = -f(q^2, q'^2, q \cdot q', -q \cdot P), \quad \text{for } f = C, \tilde{C}, D, \tilde{D}. \tag{12}$$

Due to gauge invariance,

$$q_\mu M_B^{\mu\nu} = 0, \quad M_B^{\mu\nu} q'_\nu = 0, \tag{13}$$

the scalar functions of Eq. (8) are not independent, i.e., they are related by a homogeneous set of five independent linear equations [15]. The constraints imposed by gauge invariance can be solved order by order in  $k$ , where  $k$  refers to either of  $q$  or  $q'$ . This was done in [15], where the structure-dependent part up to  $\mathcal{O}(k^4)$  was parametrized in terms of 11 low-energy coefficients, based on Lorentz invariance, gauge invariance, crossing symmetry and the discrete symmetries. Alternatively, a method suggested by Bardeen and Tung [20] may be applied to construct independent invariant amplitudes which are free from both kinematic singularities and zeros. In [16] it was pointed out that this method requires a slight generalization when applied to the VCS case where both photons are virtual. Here we will make use of the results of [16], where it was shown that  $M^{\mu\nu}$  and thus, of course,  $M_B^{\mu\nu}$  can be written as

$$\begin{aligned}
M_B^{\mu\nu} = & T_1^{\mu\nu} B_1 + T_2^{\mu\nu} \left[ B_2 - q^2 q'^2 \frac{B_6}{q \cdot q'} \right] + T_3^{\mu\nu} \left[ B_3 + (q \cdot P)^2 \frac{B_6}{q \cdot q'} \right] \\
& + T_4^{\mu\nu} \left[ B_4 - \frac{1}{2} q \cdot P (q^2 + q'^2) \frac{B_6}{q \cdot q'} \right] + T_5^{\mu\nu} \left[ B_5 + \frac{1}{2} q \cdot P (q^2 - q'^2) \frac{B_6}{q \cdot q'} \right],
\end{aligned} \tag{14}$$

with

$$\begin{aligned}
T_1^{\mu\nu} &= q'^\mu q^\nu - q \cdot q' g^{\mu\nu}, \\
T_2^{\mu\nu} &= q \cdot P (P^\mu q^\nu + q'^\mu P^\nu) - q \cdot q' P^\mu P^\nu - (q \cdot P)^2 g^{\mu\nu}, \\
T_3^{\mu\nu} &= q^2 q'^2 g^{\mu\nu} + q \cdot q' q^\mu q'^\nu - q'^2 q^\mu q^\nu - q^2 q'^\mu q'^\nu, \\
T_4^{\mu\nu} &= q \cdot P (q^2 + q'^2) g^{\mu\nu} - q \cdot P (q^\mu q^\nu + q'^\mu q'^\nu) - q'^2 P^\mu q^\nu - q^2 q'^\mu P^\nu + q \cdot q' (P^\mu q'^\nu + q^\mu P^\nu), \\
T_5^{\mu\nu} &= q \cdot P (q^2 - q'^2) g^{\mu\nu} - q \cdot P (q^\mu q^\nu - q'^\mu q'^\nu) + q'^2 P^\mu q^\nu - q^2 q'^\mu P^\nu - q \cdot q' (P^\mu q'^\nu - q^\mu P^\nu).
\end{aligned} \tag{15}$$

The functions  $B_i$  depend on the usual scalar variables and satisfy the following properties:

$$B_i(q^2, q'^2, q \cdot q', q \cdot P) = \pm B_i(q'^2, q^2, q \cdot q', -q \cdot P), \quad + : i = 1, 2, 3, 5, 6, \quad - : i = 4, \tag{16}$$

$$B_i(q^2, q'^2, q \cdot q', q \cdot P) = \pm B_i(q^2, q'^2, q \cdot q', -q \cdot P), \quad + : i = 1, 2, 3, 6, \quad - : i = 4, 5. \tag{17}$$

Each element of the tensorial basis of Eq. (15) is by construction gauge invariant. The basis is not “minimal” in the sense that the scalar functions multiplying the tensorial structures still contain kinematical singularities. In [16] it was shown that it is impossible to construct such a “minimal” basis. However, when Eq. (14) is multiplied out, the  $1/q \cdot q'$  singularities disappear, and the result reduces to the low-energy expression of [15].

### III. APPLICATION

Let us now turn to the VCS contribution to the process  $e^- + \pi^+ \rightarrow e^- + \pi^+ + \gamma$ , where the virtual photon generated by the leptonic transition current is space-like,  $q^2 < 0$ , and the final photon is real,  $q'^2 = 0$ ,  $q' \cdot \epsilon' = 0$ . The virtual Compton scattering tensor for this situation thus reduces to

$$M_B^{\mu\nu} = [q'^\mu q^\nu - q \cdot q' g^{\mu\nu}] f_1 + [q \cdot P (P^\mu q^\nu + q'^\mu P^\nu) - q \cdot q' P^\mu P^\nu - (q \cdot P)^2 g^{\mu\nu}] f_2 + [q \cdot P q^2 g^{\mu\nu} - q \cdot P q^\mu q^\nu - q^2 q'^\mu P^\nu + q \cdot q' q^\mu P^\nu] f_3, \quad (18)$$

where the functions  $f_i$  are related to the functions  $B_i$  through

$$\begin{aligned} f_1(q^2, q \cdot q', q \cdot P) &= B_1(q^2, 0, q \cdot q', q \cdot P), \\ f_2(q^2, q \cdot q', q \cdot P) &= B_2(q^2, 0, q \cdot q', q \cdot P), \\ f_3(q^2, q \cdot q', q \cdot P) &= B_4(q^2, 0, q \cdot q', q \cdot P) + B_5(q^2, 0, q \cdot q', q \cdot P). \end{aligned} \quad (19)$$

Note that for the case of at least one real photon, the terms of Eq. (14) proportional to  $B_6/q \cdot q'$  precisely cancel.

Here, we are not interested in the Bethe-Heitler contribution, where the real photon is radiated off the initial or final electron. Due to current conservation at the leptonic vertex, the polarization vector of the virtual photon can be written as  $\epsilon_\mu = e\bar{u}\gamma_\mu u/q^2$ , where  $u$  and  $\bar{u}$  refer to the Dirac spinors of the initial and final electron, respectively. We describe the reaction in the photon-pion center-of-mass system,  $\vec{p} = -\vec{q}$  and  $\vec{p}' = -\vec{q}'$ , and we choose the three-momentum transfer of the initial photon to be along the  $z$  axis,  $\vec{q} = |\vec{q}| \hat{e}_z$ . All kinematical quantities can be expressed in terms of  $\omega' = |\vec{q}'|$ ,  $\bar{q} \equiv |\vec{q}|$ , and  $z \equiv \cos(\theta) = \hat{q} \cdot \hat{q}'$ . Using gauge invariance of the hadronic VCS tensor, the invariant amplitude of Eq. (1) can be rewritten as

$$\mathcal{M} = 2Mie^2 \left( \vec{\epsilon}_T \cdot \vec{M}_T + \frac{q^2}{q_0^2} \epsilon_z M_z \right). \quad (20)$$

Choosing the Coulomb gauge for the final real photon,  $\epsilon'^\mu = (0, \vec{\epsilon}')$ , which implies  $\vec{\epsilon}' \cdot \vec{q}' = 0$ , the transverse and longitudinal parts of  $\mathcal{M}$  can be described in terms of two functions  $A_1$ ,  $A_2$  and one function  $A_9$ , respectively,

$$\vec{\epsilon}_T \cdot \vec{M}_T = \vec{\epsilon}_T \cdot \vec{\epsilon}'^* (A_1 + zA_2) - \hat{q} \times \vec{\epsilon}_T \cdot \hat{q}' \times \vec{\epsilon}'^* A_2, \quad (21)$$

$$\epsilon_z M_z = \epsilon_z \vec{\epsilon}'^* \cdot \hat{q} A_9, \quad (22)$$

where we have used the convention and nomenclature of [14].

We now contract the parametrization of Eq. (18) with  $\epsilon_\mu$  and  $\epsilon_\nu^{*\prime}$ , make use of Eq. (20), and expand the result for  $\mathcal{M}_B$  up to and including terms of order  $\omega'^2$ . In order to keep the result as transparent as possible, we do this in two steps. We first expand the kinematical factors of the tensorial basis in terms of  $\omega'$ , still keeping the functions  $f_i$  with their full set of arguments. The class  $B$  contribution to the functions  $A_1$ ,  $A_2$ , and  $A_9$  then reads

$$A_1 + zA_2 = -\omega'[(\omega_0 + \omega')f_1 + \omega'(4M^2 - 4M\omega_0 + \omega_0^2 - z\omega_0\bar{q})f_2 + 2M\bar{q}^2f_3] + \mathcal{O}(\omega'^3), \quad (23)$$

$$A_2 = -\omega'\bar{q}[f_1 - \omega'(4M - \omega_0 + z\bar{q})f_2 + 2M\omega_0f_3] + \mathcal{O}(\omega'^3), \quad (24)$$

$$A_9 = -\omega'\{(\omega_0 + \omega')f_1 + [-2M\bar{q}^2 + \omega'(4M^2 - \omega_0^2 - z\omega_0\bar{q})]f_2\} + \mathcal{O}(\omega'^3), \quad (25)$$

where  $\omega_0 \equiv q_0|_{\omega'=0} = M - \sqrt{M^2 + \bar{q}^2}$  corresponds to the energy of the initial virtual photon in the limit of zero energy of the final real photon. In Eqs. (23) - (25) we already made use of the fact that  $f_3$ , in an expansion in  $\omega'$ , is of  $\mathcal{O}(\omega')$ . This property results from the definition of  $f_3$ , Eq. (19), in terms of  $B_4$  and  $B_5$  which, in accord with their charge conjugation properties, Eq. (17), are odd functions of  $q \cdot P = q' \cdot P$ , and thus must start at least as  $\omega'$ . For this statement to be true it is crucial that we have already separated the dynamical singularities in the class  $A$  contribution.

In the next step we also expand the functions  $f_i$  in terms of  $\omega'$ , where we can restrict ourselves to first order in  $\omega'$  since the expansion of the tensorial basis has already resulted in terms which are at least of order  $\omega'$ . The relevant expansions read

$$f_i(q^2, q \cdot q', q \cdot P) = f_i(\omega_0^2 - \bar{q}^2, 0, 0) + 2\omega'\omega_0 f_{i,1}(\omega_0^2 - \bar{q}^2, 0, 0) + \omega'(\omega_0 - z\bar{q})f_{i,2}(\omega_0^2 - \bar{q}^2, 0, 0) + \omega'(2M - \omega_0 + z\bar{q})f_{i,3}(\omega_0^2 - \bar{q}^2, 0, 0) + \mathcal{O}(\omega'^2), \quad (26)$$

where  $f_{i,j}$  denotes the first partial derivative of  $f_i$  with respect to the  $j$ th argument, i.e.,  $f_{i,1}(q^2, q \cdot q', q \cdot P) = \frac{\partial}{\partial q^2}f_i(q^2, q \cdot q', q \cdot P)$  etc. Our final result for the expansion of  $\mathcal{M}_B$  to second order in  $\omega'$  is

$$A_1 + zA_2 = -\omega'\left\{\omega_0f_1 + \omega'[f_1 + 2\omega_0^2f_{1,1} + \omega_0(\omega_0 - z\bar{q})f_{1,2} + (4M^2 - 4M\omega_0 + \omega_0^2 - z\omega_0\bar{q})f_2 + 2M\bar{q}^2(2M - \omega_0 + z\bar{q})f_{3,3}]\right\} + \mathcal{O}(\omega'^3), \quad (27)$$

$$A_2 = -\omega'\bar{q}\{f_1 + \omega'[2\omega_0f_{1,1} + (\omega_0 - z\bar{q})f_{1,2} - (4M - \omega_0 + z\bar{q})f_2 + 2M\omega_0(2M - \omega_0 + z\bar{q})f_{3,3}]\} + \mathcal{O}(\omega'^3), \quad (28)$$

$$A_9 = -\omega'\left\{\omega_0f_1 - 2M\bar{q}^2f_2 + \omega'[f_1 + 2\omega_0^2f_{1,1} + \omega_0(\omega_0 - z\bar{q})f_{1,2} + (4M^2 - \omega_0^2 - z\omega_0\bar{q})f_2 - 4M\omega_0\bar{q}^2f_{2,1} - 2M\bar{q}^2(\omega_0 - z\bar{q})f_{2,2}]\right\} + \mathcal{O}(\omega'^3), \quad (29)$$

where the arguments of the functions  $f_i$  and  $f_{i,j}$  are taken to be  $(\omega_0^2 - \bar{q}^2, 0, 0)$ . When expanding the functions  $f_i$  we explicitly made use of the consequences of charge-conjugation symmetry, namely,  $f_1$  and  $f_2$  are even functions of  $q \cdot P$  and  $f_3$  is odd which follows from Eqs. (17) and (19).

#### IV. DISCUSSION

Eqs. (27) - (29) contain the central result of this work and serve as the starting point for discussing various low-energy approximations. To be specific, we will consider the multipole

expansion of [9], comment on the limit of real Compton scattering and, finally, compare the result of a  $1/M$  expansion with the parametrization of [15]. In order to fully appreciate the different expansion schemes it is useful to first discuss the kinematics of  $e^- + \pi^+ \rightarrow e^- + \pi^+ + \gamma$  in the  $\omega'$ - $\bar{q}$ -plane.

### A. Kinematical considerations

Fig. 1 shows that region of the  $\omega'$ - $\bar{q}$ -plane which is accessible to electron-scattering kinematics. Using energy conservation in the center-of-mass frame and  $|\omega| < \bar{q}$ , one obtains

$$\omega' + \sqrt{M^2 + \omega'^2} = \omega + \sqrt{M^2 + \bar{q}^2} < \bar{q} + \sqrt{M^2 + \bar{q}^2}, \quad (30)$$

and thus  $\omega' < \bar{q}$ . The diagonal  $\omega' = \bar{q}$  corresponds to the case of real Compton scattering.

Let us first consider a low-energy expansion in terms of  $\omega'$  and  $\bar{q}$  as simultaneous expansion parameters which, for example, would be a natural expansion scheme in the framework of chiral perturbation theory. In general, such an expansion is applied when  $\omega'$  and  $\bar{q}$  are smaller than a characteristic energy  $\omega_c$  of the model or theory in question. This characteristic energy is associated with either the energy gap to the first particle-production threshold or the excitation energy of the lowest excited state above the ground state and, thus, sets an upper limit to the convergence radius of the low-energy expansion. For example, in VCS off the nucleon  $\omega_c$  is equal to the pion mass  $m_\pi$ . In Fig. 1 the grey area denotes the region of the  $\omega'$ - $\bar{q}$ -plane where such a low-energy expansion is expected to converge. Clearly, if the expansion is truncated at a certain order, the domain where it is expected to give a reasonable description of the full amplitude is smaller. This regime is symbolically indicated by the black area of Fig. 1.

The multipole expansion of [9] is restricted to first order in the energy of the real photon which implies that  $\omega'$  has to be small compared with  $\omega_c$  but, in principle, no restrictions apply to  $\bar{q}$ . In particular, it is expected to work for large  $\bar{q}$ . However, when  $\bar{q}$  is of the same order of magnitude as  $\omega'$ , this scheme cannot be expected to provide an adequate parametrization of the VCS amplitude, because terms beyond the linear order in  $\omega'$  are likely to be equally important as the higher-order terms in  $\bar{q}$  included in the multipole expansion. This can be seen, e.g., for the term proportional to  $f_1$  in Eq. (27), as soon as  $\omega'$  is of the same order as the absolute value of  $\omega_0$ . In Fig. 1 the cross-hatched area schematically denotes the domain of application of the expansion of Guichon *et al.* However, one has to keep in mind that it is difficult to decide which value of  $\bar{q}$  is sufficiently large without an explicit model calculation.

### B. Multipole expansion and generalized polarizabilities

We now turn to a comparison of Eqs. (27) - (29) with the corresponding low-energy expansion in terms of generalized polarizabilities as introduced by Guichon *et al.* [9]. These

authors truncated the expansion at first order in  $\omega'$ : <sup>1</sup>

$$A_1 + zA_2 = \omega' \sqrt{\frac{E}{M}} \left[ -\sqrt{\frac{3}{2}} \omega_0 P^{(01,01)0}(\bar{q}) - \frac{3}{2} \bar{q}^2 \hat{P}^{(01,1)0}(\bar{q}) \right] + \mathcal{O}(\omega'^2), \quad (31)$$

$$A_2 = \omega' \sqrt{\frac{E}{M}} \sqrt{\frac{3}{8}} \bar{q} P^{(11,11)0}(\bar{q}) + \mathcal{O}(\omega'^2), \quad (32)$$

$$A_9 = -\omega' \sqrt{\frac{E}{M}} \sqrt{\frac{3}{2}} \omega_0 P^{(01,01)0}(\bar{q}) + \mathcal{O}(\omega'^2), \quad (33)$$

where  $E$  denotes the energy of the initial pion. Up to normalization factors, the quantities  $P^{(01,01)0}$  and  $P^{(11,11)0}$  are generalizations of the electric and magnetic polarizabilities of real Compton scattering (see, e.g., [21]) to the virtual photon case:

$$\alpha(\bar{q}) = -\frac{e^2}{4\pi} \sqrt{\frac{3}{2}} P^{(01,01)0}(\bar{q}), \quad \beta(\bar{q}) = -\frac{e^2}{4\pi} \sqrt{\frac{3}{8}} P^{(11,11)0}(\bar{q}). \quad (34)$$

The third scalar polarizability  $\hat{P}^{(01,1)0}$  expresses, to lowest order in  $\omega'$ , the difference between the charge multipole and the electric multipole.

Comparing the two low-energy expansions of Eqs. (27) - (29) and (31) - (33), we obtain the relations

$$\alpha(\bar{q}) = \frac{e^2}{4\pi} \sqrt{\frac{M}{E}} \left[ -f_1(\omega_0^2 - \bar{q}^2, 0, 0) + 2M \frac{\bar{q}^2}{\omega_0} f_2(\omega_0^2 - \bar{q}^2, 0, 0) \right], \quad (35)$$

$$\beta(\bar{q}) = \frac{e^2}{4\pi} \sqrt{\frac{M}{E}} f_1(\omega_0^2 - \bar{q}^2, 0, 0), \quad (36)$$

$$\frac{e^2}{4\pi} \hat{P}^{(01,1)0}(\bar{q}) = \frac{e^2}{4\pi} \sqrt{\frac{M}{E}} \frac{4}{3} M f_2(\omega_0^2 - \bar{q}^2, 0, 0). \quad (37)$$

From Eqs. (35) - (37), it is now evident that one of the three polarizabilities may be written as a linear combination of the remaining two. For instance, we can eliminate  $\hat{P}^{(01,1)0}$  in favor of  $\alpha(\bar{q})$  and  $\beta(\bar{q})$ ,

$$\frac{e^2}{4\pi} \hat{P}^{(01,1)0}(\bar{q}) = \frac{2\omega_0}{3\bar{q}^2} [\alpha(\bar{q}) + \beta(\bar{q})], \quad (38)$$

which is exactly the relation that has been found within the framework of the linear  $\sigma$  model [13]. We stress that this result is due to the constraint of Eq. (6), and therefore ultimately follows from the symmetry with respect to charge conjugation and pion crossing. In the multipole expansion of [9] no use has been made of this symmetry. To be specific, without this constraint the function  $f_3$  would appear in the transverse amplitudes, Eqs. (27) and (28), already at linear order in  $\omega'$ , as can be seen from Eqs. (23) and (24), resulting in one

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<sup>1</sup>For details about the notation and the definition of the generalized polarizabilities we refer the reader to [9].

additional independent function. In the framework of [15], this would correspond to the term proportional to  $\tilde{e}_1$ , indicating a violation of charge-conjugation or time-reversal symmetry.

The most surprising consequence of Eq. (38) concerns the low-energy behavior of the spin-independent electric multipole  $H^{(21,21)0}(\omega', \bar{q})$ , describing electric dipole radiation in both the initial and final states. Using Eq. (38) one gets

$$H^{(21,21)0}(\omega', \bar{q}) = \frac{4\pi}{e^2} \sqrt{\frac{8}{3}} \omega' \omega_0 \beta(\bar{q}) + \mathcal{O}(\omega'^2), \quad (39)$$

i.e., to lowest order in  $\omega'$  the electric multipole, for all  $\bar{q}$ , is given by the generalized magnetic polarizability. Since  $\omega_0 \approx -\bar{q}^2/2M$ , the right-hand side of Eq. (39) vanishes in the static limit,  $M \rightarrow \infty$ . Therefore the relation between  $H^{(21,21)0}$  and  $\beta$  is a recoil effect and not due to an intrinsic property of the target. Nevertheless, it is interesting to note that the magnetic polarizability determines the recoil contribution of an electric multipole, even though, after all, it might not be so surprising, since it is well-known that electric and magnetic effects mix when transforming from one frame to another.

Finally, we emphasize that, as a result of Eq. (38), to lowest order in  $\omega'$  both transverse amplitudes, Eqs. (27) and (28), are completely given in terms of the magnetic polarizability. The electric polarizability  $\alpha$ , as defined in Eq. (35), is part of the  $\omega'^2$  contribution to the amplitude  $A_1 + zA_2$ , which can be seen by making use of the identity  $\bar{q}^2 = \omega_0^2 - 2M\omega_0$ . However, since at the same order there are other independent contributions in Eq. (27),  $\alpha(\bar{q})$  cannot be determined from this amplitude. Thus, contrary to real Compton scattering, in VCS it is impossible to extract the generalized electric polarizability from the transverse amplitude, and one has to resort to the longitudinal amplitude  $A_9$  in order to obtain  $\alpha(\bar{q})$ .

### C. Real Compton scattering

We now take the limit of real Compton scattering (RCS) in Eqs. (27) and (28),  $\omega \equiv \bar{q} = \omega'$ , considering terms up to second order in  $\omega$ . Of course, the contribution of the longitudinal amplitude to the invariant matrix element vanishes. Making use of the expansion  $\omega_0 = -\omega^2/2M + \mathcal{O}(\omega^4)$ , we obtain

$$A_1^{RCS} + zA_2^{RCS} = -\omega^2 [f_1(0, 0, 0) + 4M^2 f_2(0, 0, 0)] + \mathcal{O}(\omega^4) = \frac{4\pi}{e^2} \omega^2 \alpha(0) + \mathcal{O}(\omega^4), \quad (40)$$

$$A_2^{RCS} = -\omega^2 f_1(0, 0, 0) + \mathcal{O}(\omega^4) = -\frac{4\pi}{e^2} \omega^2 \beta(0) + \mathcal{O}(\omega^4), \quad (41)$$

leading to the correct low-energy behavior of the RCS amplitudes [21]. We stress that in order to obtain this result, it is mandatory to keep the terms quadratic in  $\omega'$  in Eqs. (27) and (28). These terms are beyond the accuracy of the multipole expansion of [9].

### D. Low-energy expansion

In [15] the structure-dependent class- $B$  contribution was parametrized up to and including terms of fourth order in  $q$  and  $q'$ . Recently, the corresponding structure coefficients

for VCS off the nucleon have been calculated within the framework of heavy-baryon chiral perturbation theory to third order in the momenta [14], by expanding the invariant amplitude in terms of  $\omega'$  and  $\bar{q}$  simultaneously. Our general expansion in Eqs. (27) - (29) can be compared with a heavy-baryon calculation if we expand Eqs. (27) - (29) in terms of  $\bar{q}^2$  and neglect all but the leading terms of a  $1/M$  expansion. In our final result we only list the terms to quadratic order in  $\omega'$  and to quartic order in  $r$ , where  $r \in \{\omega', \bar{q}\}$ :

$$A_1^{HB} + zA_2^{HB} = -\omega'^2 \left\{ f_1(0, 0, 0) + 4M^2 f_2(0, 0, 0) \right. \\ \left. - \bar{q}^2 [f_{1,1}(0, 0, 0) + 4M^2 f_{2,1}(0, 0, 0) - 4M^2 f_{3,3}(0, 0, 0)] \right\} + \mathcal{O}(\omega'^3), \quad (42)$$

$$A_2^{HB} = -\omega' \bar{q} \left\{ f_1(0, 0, 0) - \bar{q}^2 f_{1,1}(0, 0, 0) - \omega' \bar{q} z f_{1,2}(0, 0, 0) \right\} + \mathcal{O}(\omega'^3), \quad (43)$$

$$A_9^{HB} = -\omega'^2 \left\{ f_1(0, 0, 0) + 4M^2 f_2(0, 0, 0) \right. \\ \left. - \bar{q}^2 [f_{1,1}(0, 0, 0) + 4M^2 f_{2,1}(0, 0, 0)] \right\} + \mathcal{O}(\omega'^3). \quad (44)$$

We find the following identities for the structure constants defined in [15]:

$$g_0 = f_1(0, 0, 0), \\ \tilde{c}_1 = \frac{1}{2} f_2(0, 0, 0), \\ g_{2a} = f_{1,2}(0, 0, 0), \\ g_{2b} = f_{1,1}(0, 0, 0), \\ c_3 = \frac{1}{4} f_{3,3}(0, 0, 0), \\ \tilde{c}_{3b} = \frac{1}{2} f_{2,1}(0, 0, 0) - \frac{1}{4} f_{3,3}(0, 0, 0). \quad (45)$$

The remaining three structure constants of [15] involve terms of  $\mathcal{O}(\omega'^3 \bar{q})$  and  $\mathcal{O}(\omega'^4)$  and, thus, cannot be related to the functions  $f_i$  by means of Eqs. (27) - (29). Furthermore, the presence of the  $f_{3,3}$  piece in Eq. (42) makes it impossible to extract the derivative  $\frac{d}{d\bar{q}^2} \alpha(\bar{q} = 0)$  from the  $\omega'^2 \bar{q}^2$  term. However, in the longitudinal part of the amplitude, the  $f_{3,3}$  piece is absent and the coefficients of the  $\omega'^2 \bar{q}^2$  term add up to the slope of the electric polarizability with respect to  $\bar{q}^2$  (see the discussion at the end of subsection IV B).

## V. SUMMARY AND CONCLUSION

We discussed the general amplitude for VCS off a spinless target. The results may also be applied to the spin-averaged amplitude of the nucleon case. We restricted our considerations to the matrix element involving a spacelike virtual photon in the initial state and a real photon in the final state which can be expressed in terms of one longitudinal and two transverse amplitudes. We assumed that the general matrix element may be separated into a pole contribution and a residual part which is regular as either of the two photon four-momenta approaches zero. We then discussed a low-energy expansion of the regular amplitude up to and including terms of second order in the frequency  $\omega'$  of the final photon, without restrictions on the absolute value  $\bar{q}$  of the three-momentum of the initial virtual

photon. A multipole expansion, truncated at first order in the energy of the final photon, results in two independent functions (generalized polarizabilities) instead of three as previously claimed. This reduction is obtained as a consequence of charge-conjugation invariance in combination with pion (or nucleon) crossing. Whether charge-conjugation symmetry also leads to a reduction in the number of spin-dependent generalized polarizabilities remains to be seen. At leading order in  $\omega'$ , we found that both transverse amplitudes are determined by  $\beta(\bar{q})$ , the generalization of the magnetic polarizability of RCS to arbitrary  $\bar{q}$ . On the other hand, the generalized electric polarizability  $\alpha(\bar{q})$  appears in the longitudinal amplitude only. Even in an expansion to second order in  $\omega'$ , the generalized electric polarizability cannot be extracted from the transverse part since additional independent terms appear at the same order. Furthermore, at leading order the (E1,E1) transition matrix element is governed by the generalized magnetic polarizability and vanishes in the static limit, indicating a recoil effect. In order to obtain the standard limit of RCS involving the usual electromagnetic polarizabilities  $\alpha(0)$  and  $\beta(0)$ , it is necessary to include the terms of second order in  $\omega'$ , being so far beyond the standard analysis of VCS in terms of generalized polarizabilities. Finally, we performed a  $1/M$  expansion as used in a heavy-baryon calculation and, within that framework, established the connection between the general expression and the coefficients of a recently proposed low-energy expansion.

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## REFERENCES

- [1] See, e.g., Proceedings of the *Workshop on Virtual Compton Scattering*, Clermont-Ferrand, June 1996.
- [2] G. Audit *et al.*, CEBAF Report PR 93-050, 1993.
- [3] J. F. J. van den Brand *et al.*, CEBAF Report PR 94-011, 1994.
- [4] J. Berthot *et al.*, CEBAF Report PR 94-106, 1994.
- [5] G. Audit *et al.*, MAMI proposal *Nucleon Structure Study by Virtual Compton Scattering*, 1995.
- [6] J. Shaw, private communication.
- [7] F. E. Low, Phys. Rev. **96**, 1428 (1954).
- [8] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).
- [9] P. A. M. Guichon, G. Q. Liu, A. W. Thomas, Nucl. Phys. **A591**, 606 (1995).
- [10] S. Scherer, A. Yu. Korchin, J. H. Koch, Mainz Report MKPH-T-96-4, nucl-th/9605030.
- [11] G. Q. Liu, A. W. Thomas, and P. A. M. Guichon, Adelaide Report ADP-96-15/T218, nucl-th/9605032.
- [12] M. Vanderhaeghen, Phys. Lett. B **368**, 13 (1996).
- [13] A. Metz and D. Drechsel, Mainz Report MKPH-T-96-08; Mainz Report MKPH-T-96-17, nucl-th/9607050.
- [14] T. R. Hemmert, B. R. Holstein, G. Knöchlein, S. Scherer, Mainz Report MKPH-T-96-10, nucl-th/9608042; Mainz Report MKPH-T-96-14, nucl-th/9606051.
- [15] H. W. Fearing and S. Scherer, Mainz Report MKPH-T-96-18, nucl-th/9607056.
- [16] R. Tarrach, Nuovo Cimento **28 A**, 409 (1975).
- [17] J. Bernabéu and R. Tarrach, Ann. Phys. (N.Y.) **102**, 323 (1976).
- [18] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
- [19] G. Barton, *Introduction to Dispersion Techniques in Field Theory* (Benjamin, New York, 1965), Chap. 5.
- [20] W. A. Bardeen and W.-K. Tung, Phys. Rev. **173**, 1423 (1968).
- [21] A. I. L'vov, Int. J. Mod. Phys. **A8**, 5267 (1993).

## FIGURES

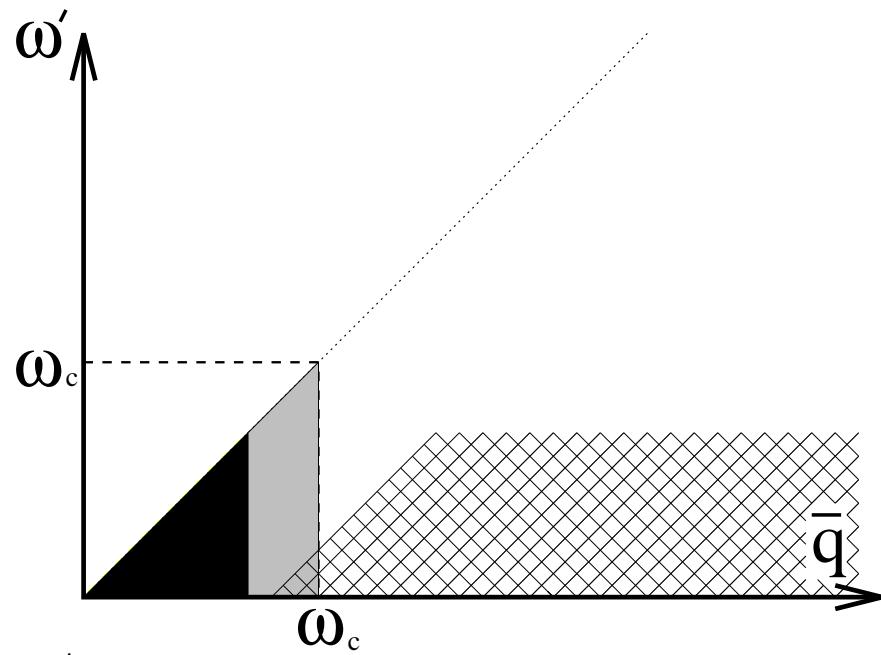


FIG. 1. The  $\omega'$ - $\bar{q}$ -plane for virtual Compton scattering with electron-scattering kinematics ( $q^2 < 0$ ,  $q'^2 = 0$ ).